THE DETERMINATION OF MINIMAL DRAG BODIES BY NEWTON'S AND BUREMANN'S DRAG LAWS

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Exact determinations of the shapes of the front parts of plane and axisymmetric minimal drag bodies in supersonic aerodynamics have been carried out only for special cases [1-4]. In this problem, the pressure on the surface of the body, which is determined by partial differential equations and boundary conditions which depend on the contour shape, is a functional of a form not known beforehand. The exact solutions mentioned were all successfully carried out using a transition, proposed by Nikol'skii [5], from the body contour to a characteristic contour which in many cases permits a reduction of the problem to a known problem in the calculus of variations. However, such a solution can be successfully constructed only under certain special relationships between the body dimensions and the free-stream Mach number M. For example, in the two-dimensional case, for each M and a certain thickness ratio which depends on M, the minimal drag body is a wedge. The body shapes for other thickness ratios are not determined. Exact solutions with restrictions on volume, surface area, etc. still are unavailable.

We note two approximate approaches. The first is related to linearization of the equations of motion, which then can be integrated and an expression obtained for the drag of the body as some functional of the body shape. In this case, the problem reduces to an ordinary problem in the calculus of variations. This procedure is applicable for determining optimal bodies for all supersonic speeds [6-9], provided the bodies so determined turn out to be thin and pointed. The second approach is based on applying the approximate formulas for pressure on a surface, obtained on the basis of elementary ideas from hypersonic flow theory ($M \gg 1$). Usually, for these purposes, the drag laws of Newton and Busemann [10, 11] are used. For bodies of the most interesting shapes with arbitrary thickness, the equation of the contour is found in final form. Further simplification for thin bodies is unnecessary, and is sometimes introduced [12] solely for the purpose of reducing the volume of computations. There are significantly more papers directed toward the second approach than toward the first. In particular, the first problem of this type was already studied by Newton [13]. However, in works along this direction, insufficient attention has been given to the fact that in the general case, the contour of the minimal drag body consists of extremal sections (two-sided extrema) and of sections of boundary extrema. The last are the boundaries of the domains of admissible variations of the parameters, and are determined both from the statement of the problem and from the regions of applicability of the approximate formulas. Ignoring this leads to certain difficulties, and to the uselessness of some solutions.

The present paper is entirely devoted to determining optimal bodies with arbitrary restrictions. We first obtain the solution using the Newtonian drag law. The necessary conditions for an extremum are obtained, as well as necessary conditions for minimum drag (the latter are sufficient for strong inequalities). As an example, we gave the solution for the case of given dimensions of the body. Some new results are obtained for plane bodies and for ducted axisymmetric bodies. For example, it turns out that the wedge is not always an optimal body. Similar investigations are then carried out for the Busemann drag law. In this case, unless the body is thin, the complete solution is known only for the case of body dimensions given. For thin bodies, the solution was recently obtained under some other restrictions [12]. In the present paper the solution is found for bodies under arbitrary restrictions. Similar investigations may be carried out for other drag laws (for example, diffuse or spectral reflection of particles in free-molecule flow).

1. We first define some notations. The isoperimetric conditions will be written in the form

$$L^j = \int_{y_0}^{y_1} f^j(y, x, x')\, dy \qquad (j = 1, \ldots, m)$$

Here x and y are rectangular coordinates; the y-axis is perpendicular to the unperturbed velocity $\mathbf{V}$; the x-axis, as well as $\mathbf{V}$ is directed from left to right and, in the axisymmetric case, coincides with the axis of symmetry; L^j are given constants; f^j are known functions; m is the number of isoperimetric conditions; $x' = dx/dy$; indices 0 and 1 denote respectively quantities at the beginning and end points of the contour, always with y_0 and x_0 considered as fixed and $x_0 = 0$.

The direction from point 0 to point 1 along a circuit around the contour is always chosen to be the positive direction. Similarly, we define the positive tangential direction. The inclination of the contour with respect to the x-axis, $\sigma = \cot^{-1} x'$, is measured counterclockwise. A corner angle on the contour is considered positive (or negative) if in passing through this corner the contour rotates counterclockwise (or clockwise).

We introduce the m-dimensional vectors $\mathbf{L} = (L^1, \ldots, L^m)$, $\mathbf{f} = (f^1, \ldots, f^m)$, $\mathbf{f}_y = (f_y{}^1, \ldots, f_y^m)$ etc. Multiplication by scalars and scalar products are defined in the usual fashion. For example, if $\boldsymbol{\lambda} = (\lambda^1, \ldots, \lambda^m)$, then

$$(\boldsymbol{\lambda}, \mathbf{f}) = \lambda^1 f^1 + \ldots + \lambda^m f^m$$

Finally, let p denote the pressure on the body surface, and let $\nu = 0$ and 1 for plane flow and axisymmetric flow, respectively.

We formulate the variational problem. Among admissible functions

$$x = x(y), \qquad x_0(y_0) = 0 \tag{1.1}$$

we seek that function which gives the minimum drag

$$\chi = (2\pi)^\nu \int_{y_0}^{y_1} p y^\nu dy \tag{1.2}$$

under the isoperimetric conditions

$$\mathbf{L} = \int_{y_0}^{y_1} \mathbf{f}(y, x, x')\, dy \tag{1.3}$$

Furthermore, we may give the body length x_1 and the thickness y_1. The expression for p and the class of admissible functions are defined by the particular form of the drag law. From the physical meaning of the quantities x and y, it follows that the functions (1.1) must be piecewise smooth.

2. Newton's drag law. According to Newtonian theory the pressure on the body surface is determined by the local inclination of the contour for $0 \leqslant \sigma \leqslant \pi$:

$$p = \rho V^2 \sin^2 \sigma = \rho V^2 (1 + x'^2)^{-1} \tag{2.1}$$

where ρ is the density of the undisturbed stream. Thus, within an unimportant multiplicative positive constant

$$\chi = \int_{y_0}^{y_1} \frac{y^\nu}{1+x'^2} dy \tag{2.2}$$

Determinations of optimal shapes using (2.1) are given in [13-24]. For unducted axisymmetric bodies, the most complete investigation was given by Lavrent'ev and Liusternik [14] and by Eggers, Resnikoff, and Dennis [15].

The class of admissible functions is determined by the region of applicability of formula (2.1). Legendre [16] already pointed out the absurdity that according to (2.1), the appearance of deep cavities or sharp edges on the body surface decreases the drag. This result can be explained by the fact that the flow pattern in those cases does not correspond to the ideas adopted in the derivation of (2.1). The latter fact is obvious if we remember that in the derivation, we have made the assumption that the gas particles offer no friction with the body surface and only lose their normal momentum there.

To exclude such cavities and spikes from consideration, (their resulting decrease in drag is due to inconsistencies with (2.1)), we may limit the class of admissible functions by the condition

$$0 \leqslant \sigma \leqslant \pi / 2 \quad \text{or} \quad 0 \leqslant x' \leqslant \infty \tag{2.3}$$

For the same reason, it is expedient to exclude from consideration curves with positive angles at corners. Thus, the class of admissible functions consists of piecewise smooth curves, issuing from the point $y = y_0$, $x = 0$, satisfying (2.3), and having no positive corner angles.

The equations of the extremal segments and the matching conditions for the different segments are found from the necessary conditions for the extrema. Segments of the boundary extremal curves may be made up of pieces of the boundary of the region (2.3)

$$x = \text{const}, \qquad y = \text{const} \tag{2.4}$$

and curves defined by the formulation of the problem. These can only be the straight lines

$$x = 0, \qquad x = x_1, \qquad y = y_0, \qquad y = y_1 \tag{2.5}$$

which are obtained if, in addition to the isoperimetric conditions, the dimensions of the body are also given; and, as easily seen, they are included in (2.4). Naturally, optimal contours contain the straight lines (2.4) only when admissible variations of the latter lead to drag increases.

To solve the problem, we set up the functional

$$I = \int_{y_0}^{y_1} \left[\frac{y^\nu}{1+x'^2} + (\lambda, \mathrm{f}) \right] dy \tag{2.6}$$

where $\lambda^1, \ldots, \lambda^m$ are Lagrange multipliers. Since, for the admissible variation, the variation of the right-hand side of (1.3) is zero, then all the variations of I and χ coincide. The contour may possess corners; the interval of integration is subdivided into a finite number of intervals in each of which x' is continuous. The value of any quantity at the ith corner will be denoted by the subscript i ($i = 2, 3, \ldots$), while subscripts $-$ and $+$ denote the values of the functions before and after the corner. Carrying out the variations and the integration by parts, we find

$$\delta I = \left[\frac{y^\nu}{1+x'^2} + (\lambda, \mathrm{f}) \right]_1 \Delta y_1 - \left[\frac{2y^\nu x'}{(1+x'^2)^2} - (\lambda, \mathrm{f}_{x'}) \right]_1 \delta x_1 +$$

$$+ \left[y^\nu \left(\frac{1}{1+x_-'^2} - \frac{1}{1+x_+'^2} \right) + (\lambda, \mathrm{f}_- - \mathrm{f}_+) \right]_i \Delta y_i - \left[\frac{2y^\nu x'}{(1+x'^2)^2} - (\lambda, \mathrm{f}_{x'}) \right]_{i-} \delta x_{i-} +$$

$$+ \left[\frac{2y^\nu x'}{(1+x'^2)^2} - (\lambda, \mathrm{f}_{x'}) \right]_{i+} \delta x_{i+} + \int_{y_0}^{y_1} \left\{ (\lambda, \mathrm{f}_x) + \frac{d}{dy} \left[\frac{2y^\nu x'}{(1+x'^2)^2} - (\lambda, \mathrm{f}_{x'}) \right] \right\} \delta x \, dy$$

where Δy_1 and Δy_i are the variations in the ordinates at the end point and at the corners. We sum over the repeated indices at all the corners. The letter δ denotes variation. We can show that at the points 1 and i

$$\delta x = \Delta x - x' \delta y - \delta x' \Delta y - x'' (\Delta y)^2 / 2 + \ldots \tag{2.7}$$

Here Δx is the displacement of the abscissa at the end or corner points, and the dots indicate terms of higher order. Using (2.7), we get

$$\begin{aligned} \delta I = {} & \left[y^\nu \frac{1+3x'^2}{(1+x'^2)^2} + (\lambda, \mathrm{f} - x'\mathrm{f}_{x'}) \right]_1 \Delta y_1 - \left[\frac{2y^\nu x'}{(1+x'^2)^2} - (\lambda, \mathrm{f}_{x'}) \right]_1 \Delta x_1 + \\ & + \left\{ y^\nu \left[\frac{1+3x_-'^2}{(1+x_-'^2)^2} - \frac{1+3x_+'^2}{(1+x_+'^2)^2} \right] + (\lambda, \mathrm{f}_- - x_-'\mathrm{f}_{x'-} - \mathrm{f}_+ + x_+'\mathrm{f}_{x'+}) \right\}_i \Delta y_i - \\ & - \left\{ 2y^\nu \left[\frac{x_-'}{(1+x_-'^2)^2} - \frac{x_+'}{(1+x_+'^2)^2} \right] - (\lambda, \mathrm{f}_{x'-} - \mathrm{f}_{x'+}) \right\}_i \Delta x_i + \\ & + \int_{y_0}^{y_1} \left\{ (\lambda, \mathrm{f}_x) + \frac{d}{dy} \left[\frac{2y^\nu x'}{(1+x'^2)^2} - (\lambda, \mathrm{f}_{x'}) \right] \right\} \delta x \, dy \end{aligned} \tag{2.8}$$

Utilizing this expression, we find, first of all, the equations of the extremals and the matching conditions. If the extremal does not agree with (2.4), then δx is arbitrary there. Consequently, on such segments, the necessary condition for an extremum (Euler equation) assumes the form

$$(\boldsymbol{\lambda}, \mathbf{f}_x) + \frac{d}{dy}\left[\frac{2y^\nu x'}{(1+x'^2)^2} - (\boldsymbol{\lambda}, \mathbf{f}_{x'})\right] = 0 \tag{2.9}$$

For many practically interesting isoperimetric conditions

$$\mathbf{f} = \mathbf{f}(y, x') \tag{2.10}$$

i.e. $\mathbf{f}_x \equiv 0$. In such a case, the equation of the extremal may be written in final form. To this end, let $q = x'$ and consider it as a parameter. In view of (2.10), we have, from (2.9)

$$\frac{2y^\nu q}{(1+q^2)^2} - (\boldsymbol{\lambda}, \mathbf{f}_{x'}) = C \qquad (C = \text{const}) \tag{2.11}$$

Here $\mathbf{f}_{x'}$ is a function of y and q. From (2.11) we find the relationship $y = y(q)$ or $q = q(y)$, and then

$$x = x(q) = \int q \frac{dy(q)}{dq} dq + C_1 \tag{2.12}$$

or

$$x = x(y) = \int q(y)\, dy + C_1 \qquad (C_1 = \text{const}) \tag{2.13}$$

Equations (2.12) and (2.13) have been derived for some concrete cases in [10,14,15,23].

We now find the matching conditions for the different sections. First of all, there arises the question whether the contour has any corners joining the extremal segments. If it has, then at these corners, in view of the arbitrariness of Δy_i and Δx_i, we have simultaneously

$$y^\nu\left[\frac{1+3x_-'^2}{(1+x_-'^2)^2} - \frac{1+3x_+'^2}{(1+x_+'^2)^2}\right] + (\boldsymbol{\lambda}, \mathbf{f}_- - x_-'\mathbf{f}_{x'-} - \mathbf{f}_+ + x_+'\mathbf{f}_{x'+}) = 0$$

$$2y^\nu\left[\frac{x_-'}{(1+x_-'^2)^2} - \frac{x_+'}{(1+x_+'^2)^2}\right] - (\boldsymbol{\lambda}, \mathbf{f}_{x'-} - \mathbf{f}_{x'+}) = 0$$

Let y, x, x' be an arbitrary point on the extremal. Write x' as x_-'. In the presence of a corner of the type in question, it is necessary for these equations to have a root $x_+' > x_-'$. The complete explanation of this can be given only for specific forms of the function $\mathbf{f}$. However, by virtue of the independence of the equations, the non-trivial roots, if any, can occur only in highly special instances. At those points where the extremals join the straight lines $x = 0$ and $y = y_1$ (the other straight lines in (2.4) are excluded because of the positive angles at the corners) only one of the equations need be satisfied. At the point of joining with the straight line $x = 0$, when $i = 2$ the quantity Δy_2 is arbitrary, and $\Delta x_2 \geqslant 0$. Thus

$$y_2^{\nu}\left[1-\frac{1+3x_+'^2}{(1+x_+'^2)^2}\right]_2+(\lambda,\ \mathbf{f}_- - \mathbf{f}_+ + x_+'\mathbf{f}_{x'+})_{y=y_2,\ x=x_-'=0}=0 \qquad (2.14)$$

$$\frac{2y_2^{\nu}x_{2+}'}{(1+x_{2+}'^2)^2}+(\lambda,\ \mathbf{f}_{x'-}-\mathbf{f}_{x'+})_{y=y_2,\ x=x_-'=0}\geqslant 0 \qquad (2.15)$$

Here the first equation (condition of Weierstrass and Erdman) is the necessary condition for an extremum, while the second is the necessary condition for a minimum. Moreover, since on the segment $x = 0$ the quantity $\delta x \geqslant 0$, then another necessary condition for a minimum is

$$\left[(\lambda,\ \mathbf{f}_x)-\frac{d}{dy}(\lambda,\ \mathbf{f}_{x'})\right]_{x\equiv x'\equiv 0}\geqslant 0 \qquad (y_0\leqslant y\leqslant y_2) \qquad (2.16)$$

Similar reasoning for the point $i = 3$, the point where the extremal joins the straight line $y = y_1$ (where Δx_3 is arbitrary, while Δy_3 and $\Delta y \leqslant 0$ for a variation of the segment $y = y_1$ independent of the variable x), leads to the necessary condition for the extremum ($x_+' = \infty$)

$$\frac{2y_1^{\nu}x_{3-}'}{(1+x_{3-}'^2)^2}-(\lambda,\ \mathbf{f}_{x'-}-\mathbf{f}_{x'+})_{y=y_1,\ x=x_3,\ x_+'=\infty}=0 \qquad (2.17)$$

and to the necessary condition for the minimum

$$y_1^{\nu}\,\frac{1+3x_{3-}'^2}{(1+x_{3-}'^2)^2}+(\lambda,\ \mathbf{f}_- - x_-'\mathbf{f}_{x'-}-\mathbf{f}_+ + x_+'\mathbf{f}_{x'+})_{y=y_1,\ x=x_3,\ x_+'=\infty}\leqslant 0 \qquad (2.18)$$

$$\left[\frac{(\lambda,\ \mathbf{f}_y)}{x'}-\frac{d}{dx}(\lambda,\ \mathbf{f}-x'\mathbf{f}_{x'})\right]_{y\equiv y_1,\ x'\equiv\infty}\leqslant 0 \qquad (x_3\leqslant x\leqslant x_1) \qquad (2.19)$$

Expression (2.18) also gives the end conditions. For a free length,

$$\frac{2y_1^{\nu}x_1'}{(1+x_1'^2)^2}-(\lambda,\ \mathbf{f}_{x'})_1=0 \qquad (\Delta x_1\neq 0) \qquad (2.20)$$

and for a free end ordinate

$$y_1^{\nu}\,\frac{1+3x_1'^2}{(1+x_1'^2)^2}+(\lambda,\ \mathbf{f}-x'\mathbf{f}_{x'})_1=0 \qquad (\Delta y_1\neq 0) \qquad (2.21)$$

A direct calculation shows that the number of conditions equals the degree of arbitrariness. In this manner, the contour of the minimum drag body may be constructed. In the general case, it may contain an end-wall (flat nose), an extremal, and the straight line $y = y_1$. In order that the constructed contour be optimal, aside from the necessary conditions on the boundaries of the segments (2.15), (2.16), (2.18) and (2.19), it is necessary to satisfy similar conditions for the extremals. To obtain these conditions, we calculate the second variation $\delta^2 I$, keeping in mind that it is necessary to consider only variations corresponding to a two-sided extremum (for boundary variations different from zero, there are already corresponding terms in δI). Moreover, we shall restrict ourselves to the case of (2.10). The expression for the second variation of the functional

$$J = \int\limits_{y_0}^{y_1} F\,(y,\, x')\; dy$$

for $x \equiv 0$ and $y_0 \leqslant y \leqslant y_2$ is of the form

$$\delta^2 J = \int\limits_{y_2}^{y_1} F_{x'x'}\,(\delta x')^2\, dy + \tfrac{1}{2}(F_{y-} - F_{y+})_2\,(\Delta y_2)^2 + \tfrac{1}{2} F_{y1}\,(\Delta y_1)^2 \quad (2.22)$$

Consequently, in the case considered, it is necessary for minimum drag that on the extremal $(y_2 \leqslant y \leqslant y_1,\ 0 \leqslant x \leqslant x_3)$

$$\frac{2y^{\nu}(3x'^2 - 1)}{(1 + x'^2)^3} + (\boldsymbol{\lambda},\, \mathbf{f}_{x'x'}) \geqslant 0 \quad (2.23)$$

at point 2

$$\frac{\nu x_{2+}'^2}{1 + x_{2+}'^2} + (\boldsymbol{\lambda},\, \mathbf{f}_{y-} - \mathbf{f}_{y+})_{2,\, x'_-=0} \geqslant 0 \quad (..24)$$

and, for a free ordinate $y_1 (\Delta y_1 \neq 0)$

$$\nu\,(1 + x_1'^2)^{-1} + (\boldsymbol{\lambda},\, \mathbf{f}_y)_1 \geqslant 0 \quad (2.25)$$

at point *1*.

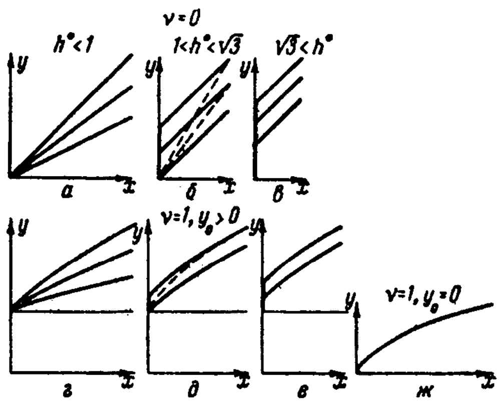

Fig. 1.

We know from the calculus of variations that the first of these inequalities (Legendre's condition) is a necessary condition for the minimum in the general case. Finally, we note that the necessary conditions for the minimum are also sufficient conditions, if strong inequalities hold in (2.15), (2.18) and (2.23) to (2.25). The solution of the variational problem may give some relative minima. In this case, we must compare the magnitudes of the drags.

As an example, let us find the optimal contours for the case of given body dimensions. We can show that the optimal contour consists of no more than two segments: an extremal and an end-wall (the latter may be absent). From (2.11), it follows that in the plane case the extremals are straight lines, while in the axisymmetric case they are convex curves (cf., e.g. [10] or [14]). In accordance with condition (2.23), the minimum corresponds only to the extremals or their sections with $x' \geqslant 3^{-1/2}$. The matching condition (2.14) gives $x_+' = 1$, i.e. $\sigma_+ = \pi/4$, independent of ν. All the possibilities of solution are shown in Fig. 1, where $h^\circ = y_1{}^\circ - y_0{}^\circ$, $y_0{}^\circ = y_0/x_1$, $y_1{}^\circ = y_1/x_1$. The plane contours with end-walls (flat noses) and the ducted axisymmetric bodies are new. We note some interesting features. For optimal bodies without end-walls, $h^\circ = h^\circ(x_0{}', y_0{}^\circ)$. For $h^\circ(1, y_0{}^\circ) \leqslant h^\circ \leqslant h^\circ(3^{-1/2}, y_0{}^\circ)$, there are two solutions – with and without end-wall (dotted line in Fig. 1), satisfying all the necessary conditions. However, the body with the end-wall has a smaller drag. The regions a, b, c of different solutions for bodies with ducts are shown in Fig. 2. All ducted bodies correspond to points on the graph lying below the straight line $y_0{}^\circ - y_1{}^\circ$. The straight lines $y_0{}^\circ = y_1{}^\circ - 1$ and $y_0{}^\circ = y_1{}^\circ - \sqrt{3}$, denoted by numbers *1* and *2*, are asymptotes of curves separating the regions a and b, and b and c, respectively. We also note that if positive angles are allowed at the corner in the plane case with $h^\circ > 1$, there would be, in addition to the contour of Fig. 1, an infinite set of optimal contours, consisting of arbitrary combinations of vertical segments and segments of straight lines with $x' = 1$. However, the drags of all such contours are the same and equal to that of the body in Fig. 1.

3. Busemann's drag law. According to Busemann's drag law, for $0 \leqslant \sigma \leqslant \pi$, the pressure on a body surface is determined only by its shape

$$p = \rho V^2 \left(\sin^2 \sigma + y^{-\nu} \frac{d\sigma}{dy} \sin \sigma \int_{y_0}^{y} y^\nu \cos \sigma \, dy \right) \tag{3.1}$$

From this, within a constant positive multiplier

$$\chi = \frac{y_1{}^{\nu+1} - y_0{}^{\nu+1}}{\nu + 1} - \cos \sigma_1 \int_{y_0}^{y_1} y^\nu \cos \sigma \, dy$$

Let us define the class of admissible functions. As in the case with Newton's formula, the restriction (2.3) remains. However, there are also other restrictions. In formula (3.1), the pressure can become negative on a convex body, which is physically meaningless, this being due to the inexact nature of the Busemann formula. The necessity for taking this into account in solving variational problems was pointed out by Hayes [11]. Thus, the class of admissible contours must be further restricted by the condition

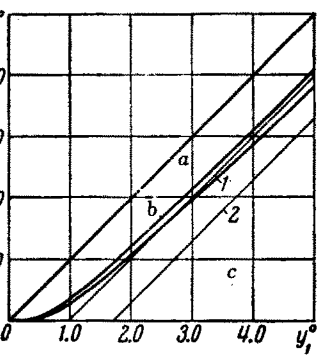

Fig. 2.

$$p \geqslant 0 \tag{3.2}$$

Finally, since the pressure cannot be infinite, we must exclude from consideration all corners, except those joining a front end-wall to the rest of the body (here $\sigma \equiv \pi/2$ and the second term in (3.1) vanishes). Thus, the class of admissible contours consists of curves, which issue from the point $y = y_0$, $x = 0$, which contain one or two smooth segments (in the latter case the first segment must be a piece of the straight line $x = 0$), and which satisfy the conditions (2.3) and (3.2).

Segments of the straight lines (2.4) and (2.5) and the curve $p = 0$ may be sections of boundary extrema. The equation of such a curve can be written in closed form and was first obtained by Lighthill [26]. In the general case, it can be obtained in the same way as in [11] and has the form

$$x = x_3 + \frac{y^{\nu+2} - (\nu+2)(y_0^{\nu+1} + \chi_3)\, y - y_3^{\nu+2} + (\nu+2)(y_0^{\nu+1} + \chi_3)\, y_3}{(\nu+1)(\nu+2) N \sin\sigma_3} \tag{3.3}$$

$$\left(N = \int_{y_0}^{y_3} y^{\nu} \cos\sigma\, dy\right)$$

Here the subscript 3 denotes quantities at the starting point of the surface $p = 0$. We can show that on the curves (3.3), we have $\sigma \geqslant 0$ and $d\sigma/dy \leqslant 0$ (the equality signs holding at $y = \infty$). An exception is the surface $p = 0$ starting from the points *0* or *2* (where $N = 0$, the equation of which is $y = \text{const}$.

Before we deal with the solution, it will be useful to clarify the order of arrangement of the different segments. The segment $y = y_1$ may only be a closing segment of the contour, and this only if on the extremal $\sigma = 0$ for $y = y_1$. This case will be considered separately. The initial portion of the contour may consist of an end-wall ($x = 0$, $y_0 \leqslant y \leqslant y_2$), and extremals. To clarify the order of alternation of the extremals and the line $p = 0$ a special investigation is necessary. For given dimensions of the body, Hayes [11] has shown on the basis of not completely rigorous arguments that the closing segment must be one with zero pressure. Subsequently a rigorous proof of this was given by Gonor [25]. For thin bodies, this fact as well as the absence of interior segments with $p = 0$ were given by Miele [12], and for a number of other isoperimetric conditions as well. In the present paper, the question of the number of zero pressure segments will not be considered. But the question of closing segments is studied for the general case.

If point *3* is the end point of the last extremal, then

$$\chi = \frac{y_3^{\nu+1} - y_0^{\nu+1}}{\nu+1} - \cos\sigma_3 \int_{y_0}^{y_3} y^{\nu} \cos\sigma\, dy$$

in which the points *3* and *1* may coincide. As before, form the functional

$$I = \frac{y_3^{\nu+1} - y_0^{\nu+1}}{\nu+1} - \cos\sigma_3 \int_{y_0}^{y_3} y^\nu \cos\sigma\, dy + \int_{y_0}^{y_1} (\Lambda, \mathbf{f})\, dy \tag{3.4}$$

Here $\Lambda^1, \ldots, \Lambda^m$ are the Lagrange multipliers. The main difference between this formula and (2.6) is in the presence of a factor before the functional which contains x_3'. Thus, in the expression for δI there appears a term with $\Delta x_3'$ (the symbol Δ as before denotes the difference in a quantity at the varied and original junction points). It is convenient to carry out the following construction. From the varied point *3* we draw the curve (3.3) until it intersects with the straight line $y = y_3$. The difference in the quantity x' at this point and at the original point *3* we denote by $\delta^\circ x_3'$. Then

$$\Delta x_3' = \delta^\circ x_3' + x_3'' \Delta y_3 = \delta^\circ x_3' + \frac{y_3^\nu}{N \sin\sigma_3} \Delta y_3 \tag{3.5}$$

where x_3' is found from (3.3).

Taking the variation of the functional (3.4), and taking into account (2.8) and (3.5), we get

$$\begin{aligned}
\delta I = {} & (\Lambda, \mathbf{f} - x'\mathbf{f}_{x'})_1 \Delta y_1 + (\Lambda, \mathbf{f}_{x'})_1 \Delta x_1 - y_3^\nu \cos\sigma_3 \sin^3\sigma_3\, \delta x_3 - \\
& - N \sin^3\sigma_3 \delta^\circ x_3' - [y^\nu \cos\sigma_3 (1 - \sin^3\sigma_+) - (\Lambda, \mathbf{f}_{x'-} - \mathbf{f}_{x'+})]_2 \Delta x_2 + \\
& + [y^\nu \cos\sigma_3 \cos^3\sigma_+ + (\Lambda, \mathbf{f}_- - \mathbf{f}_+ + x_+'\mathbf{f}_{x'+})]_2 \Delta y_2 + \\
& + \int_{y_0}^{y_3} \left\{(\Lambda, \mathbf{f}_x) + \frac{d}{dy}[y^\nu \cos\sigma_3 \sin^3\sigma - (\Lambda, \mathbf{f}_{x'})]\right\} \delta x\, dy + \int_{y_3}^{y_1} \left(\Lambda, \mathbf{f}_x - \frac{d}{dy}\mathbf{f}_{x'}\right) \delta x\, dy
\end{aligned} \tag{3.6}$$

From this we find the equation of the extremals

$$(\lambda, \mathbf{f}_x) + \frac{d}{dy}[y^\nu \sin^3\sigma - (\lambda, \mathbf{f}_{x'})] = 0 \qquad \left(\lambda = \frac{\Lambda}{\cos\sigma_3}\right) \tag{3.7}$$

and the matching condition at point *2*

$$y_2^\nu \cos^3\sigma_{2+} + (\lambda, \mathbf{f}_- - \mathbf{f}_+ + x'_+\mathbf{f}_{x'+})_{2, x = x'_- = 0} = 0 \tag{3.8}$$

Here λ are new Lagrange multipliers. When $\cos\sigma_3 = 0$ here and in all following equations, we must omit all terms not containing λ.

Equations (3.7) and (3.8) are the necessary conditions for an extremum. For the Newton drag law, the requirement on the non-negativeness of the term containing Δx_2 gave one of the necessary conditions of a boundary minimum. Here, this term is identically equal to zero. In fact, if $x_+' = x_-'$, then the coefficient of Δx_2 vanishes. Thus the necessary condition for a boundary minimum at the end-wall has the form (for $y_0 \leqslant y \leqslant y_2$, the admissible $\delta x \geqslant 0$)

$$\left\{(\boldsymbol{\lambda}, \mathbf{f}_x) + \frac{d}{dy}\,[y^\nu - (\boldsymbol{\lambda}, \mathbf{f}_{x'})]\right\}_{x\equiv x'\equiv 0} \geqslant 0 \qquad (y_0 \leqslant y \leqslant y_2) \tag{3.9}$$

At a corner the indicated term vanishes since the admissible Δx_2 vanish. Since this holds also with respect to δx, then for corners the inequality (3.9) is not required.

Let the last segment of the optimal body be an extremal (points *3* and *1* coincide). We vary this segment, in such a way that $\Delta y_1 = \Delta x_1 = \delta x_1 = 0$. From equations (3.6) and (3.7)

$$\delta I = -N \sin^3 \sigma_1 \delta^\circ x_1' \tag{3.10}$$

If $p_1 > 0$ and $\sigma_1 > 0$, then $\delta^\circ x_1'$ is arbitrary, and $\chi \equiv I$ may be changed in any direction, which contradicts the assumption of an optimal contour. Consequently, if the solution with a closing extremal segment gives $p_1 > 0$ and $\sigma_1 > 0$, then the optimal contour terminates with a segment $p = 0$.

The cases $\sigma_1 = 0$ and $\sigma_1 > 0$, $p_1 = 0$ are studied in the usual fashion and give the following end conditions:

$$y_1^\nu \sin^3 \sigma_1 - (\boldsymbol{\lambda}, \mathbf{f}_{x'})_1 = 0 \qquad \text{for } \Delta x_1 \neq 0 \tag{3.11}$$

$$y_1^\nu \sin^2 \sigma_1 \cos \sigma_1 + (\boldsymbol{\lambda}, \mathbf{f} - x'\mathbf{f}_{x'})_1 = 0 \qquad \text{for } \Delta y_1 \neq 0 \tag{3.12}$$

We note that for $\sigma_1 > 0$ and $p_1 = 0$, in accordance with (3.1), we have $\delta^\circ x_1' \leqslant 0$, i.e. a boundary extremum is realized through $\delta^\circ x_1'$.

To obtain the necessary conditions for an extremum at the points *3* and *1* in the general case ($y_1 > y_3$), it suffices to vary the contour in the class of curves with a closing segment $p = 0$. Then, in agreement with (3.3)

$$\delta x_3 = \Delta x_1 - x_1' \Delta y_1 - \frac{x_1 - x_3}{x_3'} \delta^\circ x_3' - \frac{x_1 - x_3 + (y_3 - y_1)\, x_3'}{N \cos \sigma_3} \delta\chi$$

and on the segment *31*

$$\delta x = \delta x_3 + \frac{x - x_3}{x_3'} \delta^\circ x_3' + \frac{x - x_3 + (y_3 - y)\, x_3'}{N \cos \sigma_3} \delta\chi$$

Substituting these expressions into (3.6), fixing the segment *02*, and taking into account the condition (3.7) and the fact that for an optimal contour $\delta\chi = 0$, we get

$$\delta I = [(y_3{}^{\nu} \sin^3 \sigma_3 - a)\, x_1{}' + (\boldsymbol{\lambda},\, \mathbf{f} - x'\mathbf{f}_{x'})_1] \cos \sigma_3\, \Delta y_1 + \\ + [a - y_3{}^{\nu} \sin^3 \sigma_3 + (\boldsymbol{\lambda},\, \mathbf{f}_{x'})_1] \cos \sigma_3 \Delta x_1 + [b - N \sin^2 \sigma_3 - \\ - (x_1 - x_3)\, (a - y_3{}^{\nu} \sin^3 \sigma_3)] \sin \sigma_3 \delta^{\circ} x_3{}'$$

where the variations Δy_1, Δx_1 and $\delta^{o} x_3{}'$ are independent, and

$$a = \int\limits_{y_3}^{y_1} \left(\boldsymbol{\lambda},\, \mathbf{f}_x - \frac{d}{dy}\, \mathbf{f}_{x'}\right) dy, \qquad b = \int\limits_{y_3}^{y_1} (x - x_3) \left(\boldsymbol{\lambda},\, \mathbf{f}_x - \frac{d}{dy}\, \mathbf{f}_{x'}\right) dy$$

Equating δI to zero, we find the matching condition at the point *3*

$$b - N \sin^2 \sigma_3 - (x_1 - x_3)\, (a - y_3{}^{\nu} \sin^3 \sigma_3) = 0 \tag{3.13}$$

and the end conditions

$$a - y_3{}^{\nu} \sin^3 \sigma_3 + (\boldsymbol{\lambda},\, \mathbf{f}_{x'})_1 = 0 \quad \text{for } \Delta x_1 \neq 0 \tag{3.14}$$

$$(y_3{}^{\nu} \sin^3 \sigma_3 - a)\, x_1{}' + (\boldsymbol{\lambda},\, \mathbf{f} - x'\mathbf{f}_{x'})_1 = 0 \quad \text{for } \Delta y_1 \neq 0 \tag{3.15}$$

We observe that equations (3.11) and (3.12) are special cases of the two subsequent equations when $y_3 = y_1$. The equations determine the contour of the body, and also the coordinates of the point *3*.

When (2.10) is satisfied, the equations are significantly simpler; in particular, as in the case with Newton's formula, the equations for the extremal may be written in parametric form. Equations (3.13) to (3.15) simplify to

$$(\boldsymbol{\lambda},\, \mathbf{c} - (x_1 - x)\, \mathbf{f}_{x'})_3 - N \sin^2 \sigma_3 + (x_1 - x_3)\, y_3{}^{\nu} \sin^3 \sigma_3 = 0 \quad \left(\mathbf{c} = \int\limits_{y_3}^{y_1} x'\mathbf{f}_{x'}\, dy\right) \tag{3.16}$$

$$(\boldsymbol{\lambda},\, \mathbf{f}_{x'})_3 - y_3{}^{\nu} \sin^3 \sigma_3 = 0 \quad \text{for } \Delta x_1 \neq 0 \tag{3.17}$$

$$y_3{}^{\nu} x_1{}' \sin^3 \sigma_3 + (\boldsymbol{\lambda},\, \mathbf{f}_1 - x_1{}'\mathbf{f}_{x'})_3 = 0 \quad \text{for } \Delta y_1 \neq 0 \tag{3.18}$$

We now find the necessary conditions for a minimum for the segment of the contour $y = y_1$ and for extremals. The first condition is found from the corresponding expression for δI and is identical with (2.19). To derive the remaining conditions, we consider only the case (2.10), which will restrict the variations of the ordinate of the point *2* and a portion of the extremal *23*. Using formula (2.22), we find

$$\delta^2 I = \frac{1}{2} [\nu \cos \sigma_+ + (\boldsymbol{\lambda},\, \mathbf{f}_{y-} - \mathbf{f}_{y+})]_2\, (\Delta y_2)^2 +$$

$$+\int_{y_3}^{y_1} [3y^{\nu} \cos\sigma \sin^4\sigma + (\lambda, \mathbf{f}_{x'x'})] (\delta x')^2 dy$$

Consequently, the necessary conditions for the minimum are

$$\nu \cos\sigma_{2+} + (\lambda, \mathbf{f}_{y-} - \mathbf{f}_{y+})_2 > 0 \tag{3.19}$$

$$3y^{\nu} \cos\sigma \sin^4\sigma + (\lambda, \mathbf{f}_{x'x'}) > 0 \text{ where } y_3 \leqslant y \leqslant y_1 \tag{3.20}$$

Under strong inequalities, the conditions (3.19), (3.20), (2.19) and (3.9) (the last only possible when there are no corners) are not only necessary but also sufficient conditions for minimum drag.

In calculating χ and I it was convenient to use the fact that, as long as the closing segment *31* has zero pressure, then the drag of the entire body equals that of the front segment *03*. In regard to this, however, two questions remain unclear; first, how the drag on the closing segment changes when the front segment is varied, and second, how the total drag changes when the closing segment is varied. Without entering into details, we merely indicate here that if the segment *03* is optimal, then in both cases, any admissible variation increases the total drag. The proof of this fact follows from the observation that N appears with coefficients of the same sign in expression (3.1) for the pressure on the closing segment (where $d\sigma/dy < 0$) as well as in the expression for χ.

In the previous section, optimal contours for given body dimensions were obtained using the Newtonian drag law. For comparison we carry out similar investigation for the Busemann drag formula. We note, however, that the main results for this question have already been found in [11, 24, 25]. It turns out that optimal contours consist of no more than three sections: an end wall, an extremal curve, and a zero pressure curve – arranged in that order. From equation (3.7), it follows that in the plane case the extremals are straight lines, while in the axisymmetric case they are convex curves (cf., e.g. [11, 24]). In conformity with condition (3.20), the minimum is realized for all extremals with $x' \geqslant 0$. The matching condition (3.8) reduces to the condition that there be no corner (discontinuity in slope) at the junction point between the end-wall and the extremal. Thus, in the plane case, the optimal contours contain no end-walls. Contours of axisymmetric bodies without ducts consist of three sections: an end-wall, an extremal, and a segment of zero-pressure curve. Ducted bodies may or may not have end-walls – the applicability of each configuration is determined by comparing the magnitudes of χ. From condition (3.16), it follows that $x_3 = 0.5\, x_1$ for plane bodies [11] and $x_3 > 0.6\, x_1$ for axisymmetric bodies without ducts [11, 25].

In conclusion, we make certain observations on the determination of optimal bodies in the exact formulation. Although in this case restrictions (2.3) and (3.2), related to the inexactness of the drag formulas, drop out, the other restrictions connected with the formulation of the problem remain. For given dimensions this leads to the straight lines (2.4), which may turn out to be segments of a boundary extremum. Thus one may try to construct optimal contours, assuming that they consist, for example, of segments of extremals and end-walls. In doing this, it is first of all necessary to know how to determine the extremal segment for given dimensions of the end-wall. This part of the problem was solved in [2,3]. Next, it is necessary to select such an end-wall dimension that its increase or decrease results in an increase in the drag. Such a procedure may be carried out, for example, by numerically comparing the magnitude of χ for bodies consisting of an end-wall and an extremal segment for different end-wall dimensions. However, even if such an end-wall dimension is found, it still does not indicate that the constructed contour is optimal. It is necessary that part of the end-wall be part of a boundary extremum. Clarification of this feature leads to major difficulties, the solution of which still is unclear at this point.

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